Metrics of a fixed doubling dimension and efficient approximation of intractable combinatorial problems

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Introduction

- A vast majority of combinatorial optimization problems (e.g., Set Cover, Hitting Set Problem, Maximal Clique Problem, etc.) are known to be intractable and hardly approximable in general settings
- Meanwhile, for many actual special cases of these problems there are known efficient exact or approximation algorithms
- For instance, many combinatorial optimization problems become much more approximable being formulated in geometrical setting, in the Euclidean spaces of finite dimension
- But what can we say about their approximation in metric spaces?
- In this lecture we will consider these aspects on Capacitated Vehicle Routing Problem (CVRP), the well-known generalization of the classic Traveling Salesman Problem

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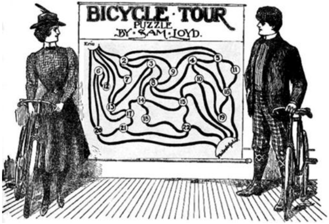
Traveling Salesman Problem (TSP)

Problem statement

Input: complete weighted graph $G = (V, E, w)$
Required: to find a Hamiltonian cycle of the minimum (or maximum) weight



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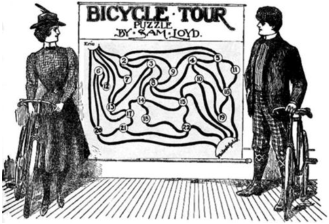

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Complexity bounds

- exhaustive search $\Theta((n-1)!)$
- dynamic programming $\Theta(n^2 2^n)$

Combinatorial optimization problems

- Combinatorial optimization problem $\mathcal{I}$

$$I: \; OPT_I = \min\{COST_I(x) \colon x \in X_I\}$$

$n := LEN(I)$ is instance length $I \in \mathcal{I}$.

- Algorithm is an arbitrary function $Alg : I \mapsto Alg(I) \in X_I$, computable in time $TIME_{Alg}(I)$
- Algorithm Alg is called polynomial time, if

$$TIME_{Alg}(I) = O(poly(LEN(I)) \; (I \in \mathcal{I})$$

- Algorithm Alg is called optimal, if

$$APP_I := COST_I(Alg(I)) = OPT_I$$

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Approximation algorithms and Schemes

- Let, for some $r = r(I)$ the equation

$$OPT_I \leq APP_I = COST_I(Alg(I)) \leq r \cdot OPT_I \quad (I \in \mathcal{I})$$

is valid. Then Alg is called r-approximation algorithm for the problem $\mathcal{I}$

- Algorithms with fixed accuracy bounds $r(I) = \text{const}$ and approximation schemes (PTAS or QPTAS) attract the most interest
- The problem $\mathcal{I}$ has PTAS (QPTAS), if for any $\varepsilon > 0$ there exists $(1+\varepsilon)$-approximation polynomial time (quasi-polynomial time) algorithm Alg_ε
- For any PTAS, time complexity bound $TIME_{Alg_\varepsilon}(I) = O(poly(LEN(I)) = O(poly(n))$ depends on n polynomially but can have an arbitrarily dependence on ε, e.g.

$$TIME_{Alg_\varepsilon}(I) = O(n^{\exp(1/\varepsilon^3)})$$

- For a QPTAS and any fixed $\varepsilon > 0$, time complexity $TIME_{Alg_\varepsilon}(I)$ is a quasi-polynomial of n, e.g.

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Some known complexity and approximation results

Complexity

- (Karp, 1972) TSP is strongly NP-hard
- (Sahni and Gonzales, 1976) TSP can not be approximated within $O(2^n)$ (unless $P = NP$)
- (Papadimitriou, 1977) Euclidean TSP is NP-hard

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Approximation

- (Christofides-Serdyukov, 1976) Metric TSP belongs to APX
- (Arora, 1996; Mitchell, 1996) First Polynomial Time Approximation Schemes (PTAS) for TSP on the plane
- (Arora, 1998) Euclidean TSP in $\mathbb{R}^d$ for any fixed $d > 1$ has EPTAS (but has no FPTAS unless $P = NP$)
- (Trevisan, 2000) TSP in the d-dimensional Euclidean space is APX-hard provided $d = \Omega(\log n)$
- (Talwar, 2004) QPTAS for the TSP in spaces of a fixed doubling dimension
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Contents

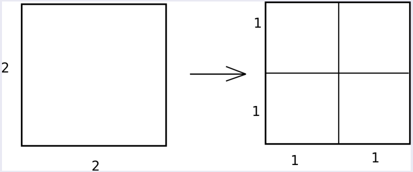
Preliminaries

Definition

A metric space (Z, ρ) has doubling dimension d, if for an arbitrary $z \in Z$ and $R > 0$, there exist $z_1, \ldots, z_t \in Z$, $t \leq 2^d$ such that

$$B(z, R) \subseteq \bigcup_{j=1}^{t} B(z_j, R/2).$$

An example: metric l_∞^d for $d = 2$



An example: metric l_2^d

Any d-dimensional Euclidean space is a metric space of doubling dimension $O(d)$.

Preliminaries

Lemma 1

Let $Z' \subset Z$ *be a non-empty subspace of a finite aspect ratio* Δ/α, *where* $\Delta = \sup\{\rho(u,v)\colon u,v \in Z'\}$ *and* $\alpha = \inf\{\rho(u,v)\colon \{u,v\} \subset Z'\}$. *Then,*

$$|Z'| \leq \left(\frac{2\Delta}{\alpha}\right)^d.$$

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$$|Z'| \leq \left(\frac{2\Delta}{\alpha}\right)^d .$$

Lemma 2

Let (Z, ρ) be a metric space of doubling dimension $d > 1$, and $U \neq \varnothing$ be an arbitrary finite subspace of radius R. Then,

$$w(\mathrm{MST}(U)) \leq 12R \cdot |U|^{1-1/d}.$$

Embeddings and snowflakes

Embedding to l_p^D

Finite metric space (Z, ρ), $|Z| = n$ is embeddable to l_p with distortion L, if there is a Lipschitz injective mapping $f: Z \to l_p$, s.t. $\rho(z_1, z_2) \leq \|f(z_1) - f(z_2)\| \leq L \cdot \rho(z_1, z_2)$ for any $z_1, z_2 \in Z$

Bourgain's Theorem, 1985

For every n-point finite metric space there exists an embedding into l_2 with distortion $O(\log n)$

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Extended this result to the case of any $1 \leq p \leq \infty$ and dimension $O(\log n)$

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Logarithmic dimension is not good (recall Trevisan's inapproximability result). We need embedding with fixed distortion into the Euclidean space of fixed dimension

Snowflakes

For any metric space (Z, ρ) and $0 < \alpha < 1$, the space (Z, ρ^α) is called a snowflake

Theorem (Assouad, Abraham et al.)

Let (Z, ρ) be a metric space of doubling dimension d, $p > 1$, $0 < \alpha < 1$, $\theta > 0$ and $2^{192/\theta} \leq k \leq d$. There exists an embedding $f\colon Z \to l_p^{\mathbf{D}}$, s.t.

$$L = O\left(k^{1+\theta} 2^{d/(pk)}/(1-\alpha)\right) \text{ and } \mathbf{D} = O\left(\frac{d2^{d/k}}{\alpha\theta}\left(1 - \frac{\log(1-\alpha)}{\log k}\right)\right).$$

Example

Illustrate the applicability of this approach on approximation of the metric CVRP

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Example

Illustrate the applicability of this approach on approximation of the metric CVRP ...

Capacitated Vehicle Routing Problem

Input

A complete weighted graph $G = (Z, E, D, w)$, where $Z = X \cup \{y\}$,

- $X = \{x_1, \ldots, x_n\}$ are customers, y is a depot
- $D \colon X \to \mathbb{Z}_+$ is a distribution of the customer demand
- $w \colon E \to \mathbb{R}_+$ specifies direct transportation costs

and an integer capacity bound $q \geq 3$.

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Routes

An ordered pair $\mathcal{R} = (\pi, S_{\mathcal{R}})$, where

- $\pi = y, x_{i_1}, \dots, x_{i_t}, y$ is a cycle in the graph G
- $S_{\mathcal{R}} \colon X \to \mathbb{Z}_+$ a distribution of the covered customer demand

is called a *route* of the cost

$$w(\mathcal{R}) = w(y, x_{i_1}) + w(x_{i_1}, x_{i_2}) + \cdots + w(x_{i_{t-1}}, x_{i_t}) + w(x_{i_t}, y).$$

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Feasible routes

The route $\mathcal{R} = (\pi, S_{\mathcal{R}})$ is called *feasible*, if

$$S_{\mathcal{R}}(x) = \begin{cases} \leq & D(x), \quad x \in \{x_{i_1}, \dots, x_{i_t}\} \\ = & 0, \quad \text{otherwise} \end{cases} \quad \text{and} \quad \sum_{x \in X} S_{\mathcal{R}}(x) \leq q.$$

Goal

To construct a set of feasible routes $\mathfrak{S} = \sum_{\mathcal{R} \in S} w(\mathcal{R})$ of the minimum transportation cost that service the total customer demand.

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The problem

$$w(\mathfrak{S}) \equiv \sum_{\mathcal{R} \in \mathfrak{S}} w(\mathcal{R}) \to \min$$

$$\sum_{\mathcal{R} \in \mathfrak{S}} S_{\mathcal{R}}(x) = D(x) \quad (x \in X).$$

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Additional constraints

(i) for some $d > 1$, the weighting function w is a metric of **doubling dimension** d;

(ii) the problem is supposed to have an optimal solution, whose number of routes does not exceed $\operatorname{polylog} n$.

Motivation

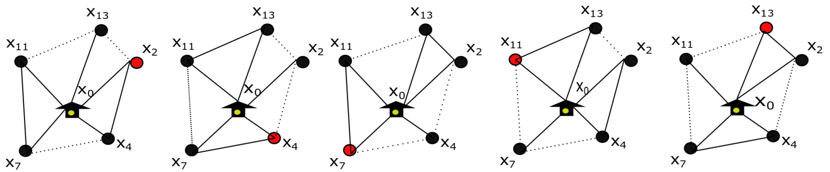
The data from survey "Road-based goods transportation: a survey of real-world logistics applications from 2000 to 2015"

The spheres of applications

- Oil, gas and fuel transportation
- Retail applications
- Waste collection and management
- Mail and Small package delivery
- Food distribution

ITP heuristic :: Main Idea

Consider the CVRP with $n = 5$ customers, $q = 2$ capacity and one time window



The ITP solution

$S_{ITP} = argmin\{w(S(x)) : x \in \{x_2, x_4, x_7, x_{11}, x_{13}\}\}$

Haimovich - Rinnooy Kan approximation scheme

1. Relabel the customers in the order $\underbrace{r_1 \geq \ldots \geq r_{k-1}}_{X_{out}} \geq \underbrace{r_k \geq \ldots \geq r_n}_{X_{in}}$, where $r_i = w(y, x_i)$
2. For the given $\varepsilon > 0$, find the minimum $\tilde{k} = \tilde{k}(\varepsilon, q, \beta)$, for which the relative error fulfils the inequality
$$e(\tilde{k}) = \frac{\mathrm{CVRP}^*(X_{out}) + \mathrm{ITP}(X_{in}) - \mathrm{CVRP}^*(X)}{\mathrm{CVRP}^*(X)} \leq \varepsilon.$$
Here $X_{out} = \{x_1, \ldots, x_{\tilde{k}-1}\}$, $X_{in} = X \setminus X_{out}$, and $\mathrm{ITP}(X_{in})$ is an upper cost bound for the partial solution obtained by ITP heuristic.
3. Find an exact solution S_{DP} for the X_{out} by dynamic programming.
4. Find an approximate solution $\mathrm{S}_{\mathrm{ITP}}$ for X_{in}.
5. Output the combined solution $S = \mathrm{S}_{\mathrm{DP}} \cup \mathrm{S}_{\mathrm{ITP}}$.

Haimovich - Rinnooy Kan approximation scheme

Lemma 3

$$\mathrm{ITP}(X) = w\left(S_{ITP}\right) \leq 2\left\lceil \frac{n}{q} \right\rceil \frac{\sum_{i=1}^{n} r_i}{n} + (1 - 1/q)\,\mathrm{TSP}^*(X).$$

Lemma 4

For any metric CVRP,

$$\mathrm{CVRP}^*(X) \geq \max\left\{\mathrm{TSP}^*(X \cup \{y\}), 2r_1, \frac{2}{q}\sum_{i=1}^{n} r_i\right\}.$$

Lemma 5

For any partition $X = X_{out} \cup X_{in}$ *and any metric* CVRP,

$$\mathrm{CVRP}^*(X_{in}) + \mathrm{CVRP}^*(X_{out}) \leq \mathrm{CVRP}^*(X) + 4(k-1)r_k.$$

Haimovich - Rinnooy Kan approximation scheme

Lemma 6

Let $X \subset B(0,R) = \{x \in l_2^D : \|x\|_2 \leq R\}$. *Then,*

$$\mathrm{TSP}^*(X) \leq C_D \cdot R + C_D^* \cdot R^{1/D} \left(\sum_{i=1}^{n} r_i \right)^{1-1/D},$$

for $C_D = D^2 \cdot 2^{D+1}$ *and* $C_D^* = 4\sqrt{2} \cdot D^{(1+3/D)/2}$.

If the instance of the CVRP s.t.

1. (Z, w), where $Z = X \cup \{y\}$ is a metric space of doubling dimension $d > 2$
2. for any $x \in X$, $a \leq w(x,y) \leq b$ for some $0 < a < b$, then

Lemma 7

$$\mathrm{TSP}_w^*(X') \leq \mathrm{TSP}_{l_2^{\mathbf{D}}}^*(X') \leq \frac{C_{\mathbf{D}} L}{\sqrt{a}} \cdot R + \frac{C_{\mathbf{D}}^* L}{\sqrt{a}} \cdot R^{1/\mathbf{D}} \left(\sum_{x_i \in X'} r_i \right)^{1-1/\mathbf{D}}.$$

Haimovich - Rinnooy Kan scheme in metric space of a fixed doubling dimension

Theorem

A $(1+\varepsilon)$-approximate solution for an arbitrary instance CVRP satisfying conditions 1 and 2 can be obtained in time

$$O\left(qk^3 2^k\right) + \mathrm{TIME}(\mathrm{TSP}, \beta, n) + O(n^2),$$

where

$$\begin{aligned} k &= k(\varepsilon, q, \beta, \mathbf{D}, a) \\ &= O\left(\left(\frac{q}{\varepsilon}\right)^{\mathbf{D}} \left(\frac{4\sqrt{2}\beta\mathbf{D}^{(1+3/\mathbf{D})/2} \cdot L}{\sqrt{a}}\right)^{\mathbf{D}} + \frac{2^{\mathbf{D}} \cdot \beta L}{\sqrt{a}}\right) \cdot \left(\exp\left(\frac{q}{\varepsilon}\right)\right)^2, \end{aligned}$$

$\mathrm{TIME}(\mathrm{TSP}, \beta, n)$ is running time of finding β-approximate solution of the auxiliary TSP instance, $\mathbf{D} = O(d \cdot \log\log d)$ and $L = \mathrm{poly}\,(d)$.

Conclusion

- for some combinatorial problems including TSP, CVRP efficient approximation results (QPTAS or even PTAS) proposed initially for the finite dimensional Euclidean spaces can be extended to metric spaces of fixed doubling dimension
- there are two approaches for such an extension, based on embeddings and snowflakes and bypassing the embedding
- the question: *'Can the QPTAS proposed by A.Das and C.Mathieu for the Euclidean CVRP be extended to metric spaces of a fixed doubling dimension without any restriction on the capacity growth?'* still remains open.

Thank you for your attention!